Day 1 - A few notes on the probability theory

Flip a coin and compute the likelihood

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UNIVERSITÉ DE SHERBROOKE

PART 1

Probability theory from scratch

Chevalier de Méré (1607-1684), Pascal (1623-1662) and Fermat (1607-1665):

A few notes on History

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- de Méré: wit / courtier (Louis XIV) brought Pascal's attention to 2 problems (1650-1655):

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2. division problem (problèmes des parties)

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- Pascal solved the hardest problem (correspondence with Fermat)

A few notes on History

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- **dice problem**:

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- **division problem**:

  "An elderly nobleman, staying at his country house, was extremely fond of watching ball games, and so he called in two young farmhands, saying, 'Here are four ducats for which you may play; the one who first takes eight games is the winner.' So they began to play, but when one had five games and the other three games, they lost the ball and were unable to finish. The question is how the prize should be divided."

Ore, O. Pascal and the Invention of Probability Theory (1960).

A few notes on History

- Reverend Thomas Bayes (1701 – 1761) *An Essay Towards Solving a Problem in the Doctrine of Chances* (published posthumously)

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- Pierre-Simon Laplace (1749–1827) *Théorie Analytique des Probabilités in 1812*

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- Huygens, de Moivre, Galton, Gauss, von Mises, Kolmogorov, Neyman, Wiener, Wald, Pearson, Shannon, Fisher

References

1. Hacking I., The Emergence of Probability (2006).

References

1. Hacking I., The Emergence of Probability (2006).
2. Hendricks V. F., Pedersen S. A., Jørgensen K. F. , Probability Theory Philosophy, Recent History and Relations to Science (2001).

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2. Hendricks V. F., Pedersen S. A., Jørgensen K. F. , Probability Theory Philosophy, Recent History and Relations to Science (2001).
3. Brémaud P., An Introduction to Probability Modelling (1988).

Examples

- Allesina and Tang, **Stability criteria for complex ecosystems**, *Nature* (2012),

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- Expending the classical TIB: short talk this pm!

Let's flip a coin

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Probability space

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   - "Head" "Tail"

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- $P(\emptyset) = 0$; P("Head") = p; P("Tail") = 1 – p; $P(\Omega) = 1$

Occurrence of species 1 on an island

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2. $\mathcal{F}$: set of events *i.e.* 0 or more outcomes
   - $\emptyset$, "0", "1", "1 or 0" (Ω)
3. P: assign a probability / map events occurrence into [0,1]
   - $P(\emptyset) = 0$; $P(''1'') = p$; $P(''0'') = 1 - p$; $P(\Omega) = 1$

Occurrence of species 1 and species 2 on an island

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1. Ω: {"00", "01", "10", "11"}

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2. $\mathcal{F}$: $\emptyset$, Ω, "01", "at least one species"

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2. $\mathcal{F}$: $\emptyset$, Ω, "01", "at least one species"
3. P: p_{00}, p_{01}, ...

Let “A” and “B” denotes two distinct events:

Combining events

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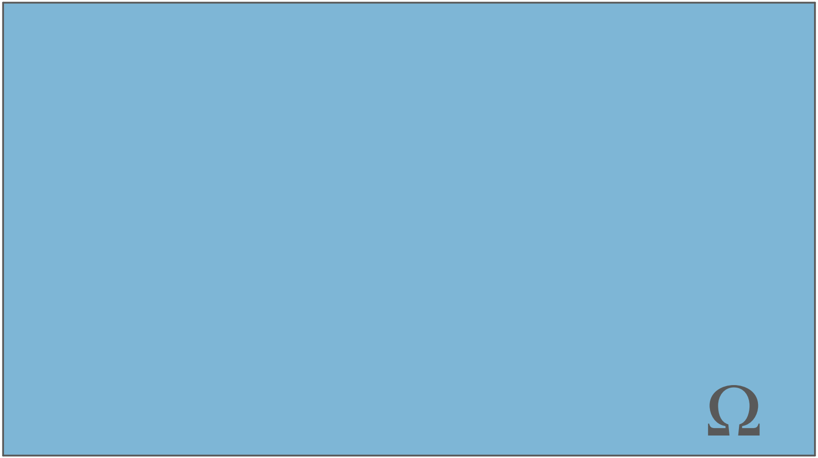
NB: $A \cup \overline{A} = \Omega$ and $A \cap \overline{A} = \emptyset$

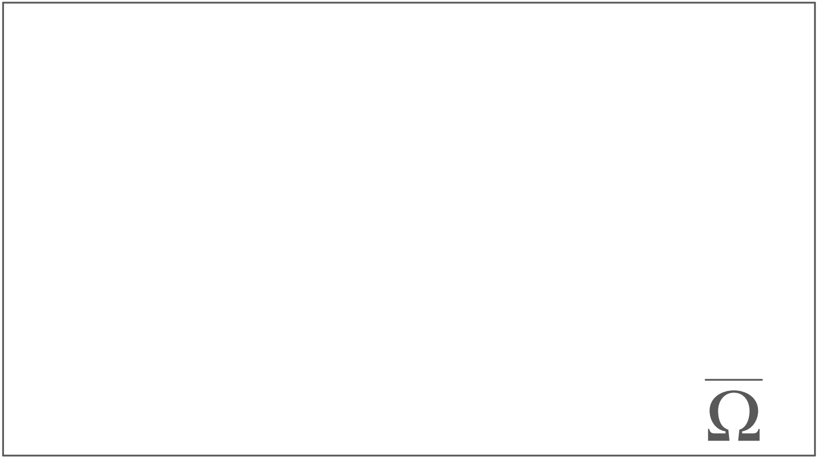
NB: $P(\overline{A}) = 1 = P(A)$

Combining events

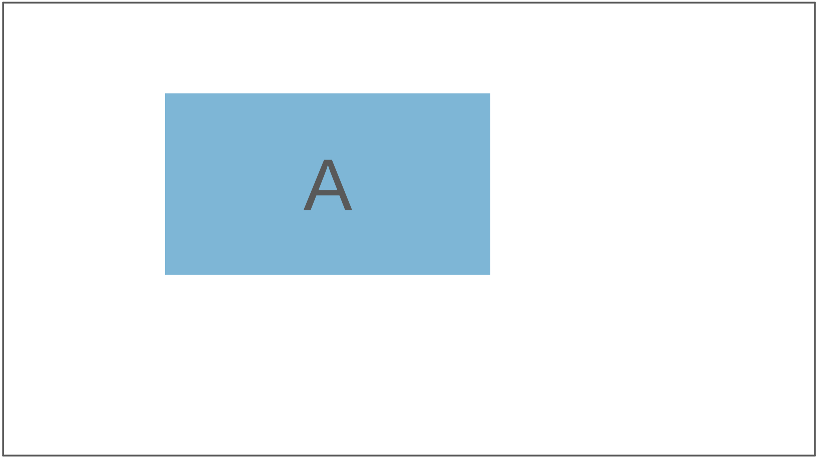


$P(\Omega) = 1$

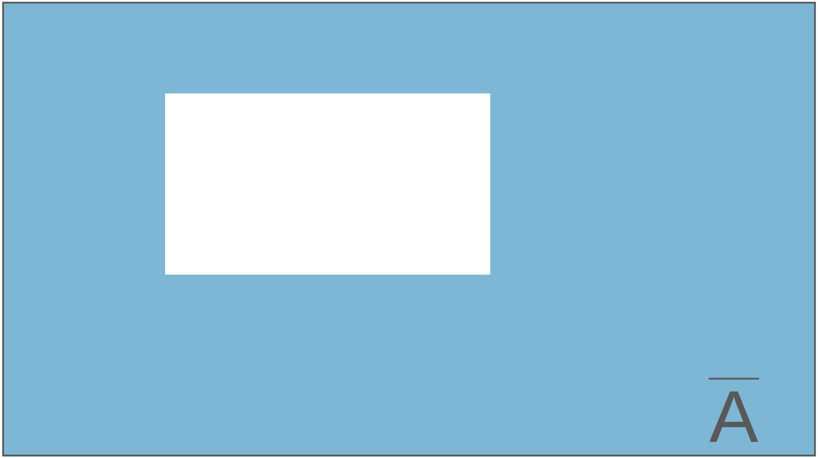
Combining events



$$P(\overline{\Omega}) = P(\emptyset) = 1 - P(\Omega) = 0$$

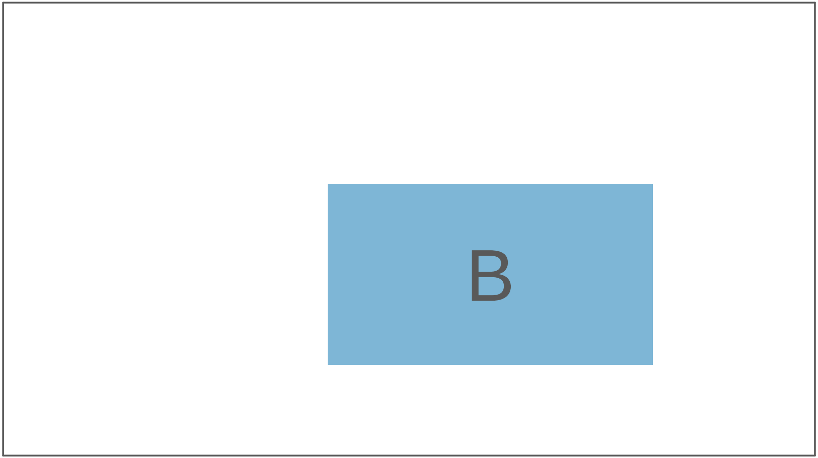


P(A)

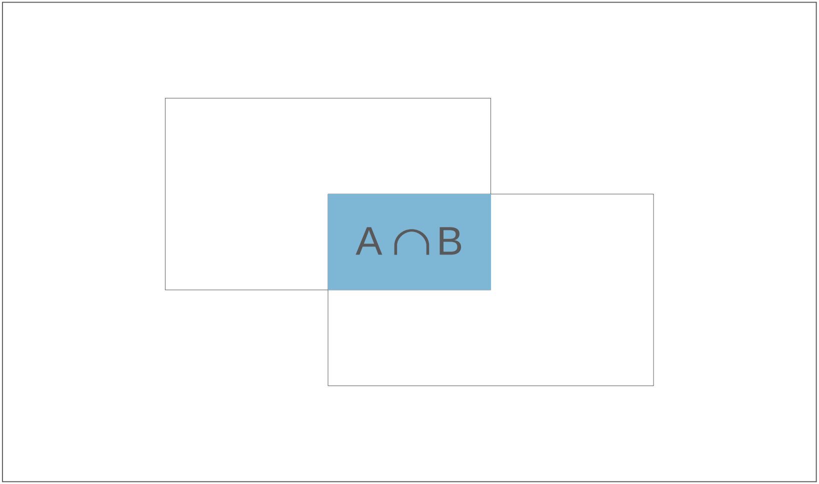


$$P(\overline{A}) = 1 - P(A)$$

Combining events

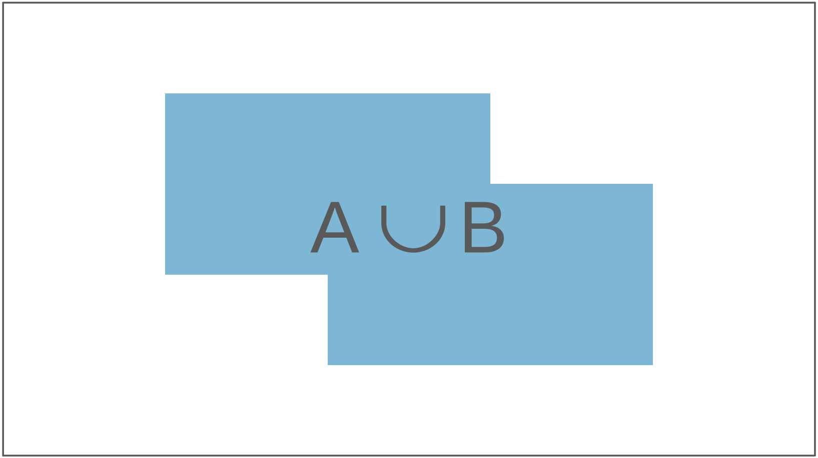


P(B)

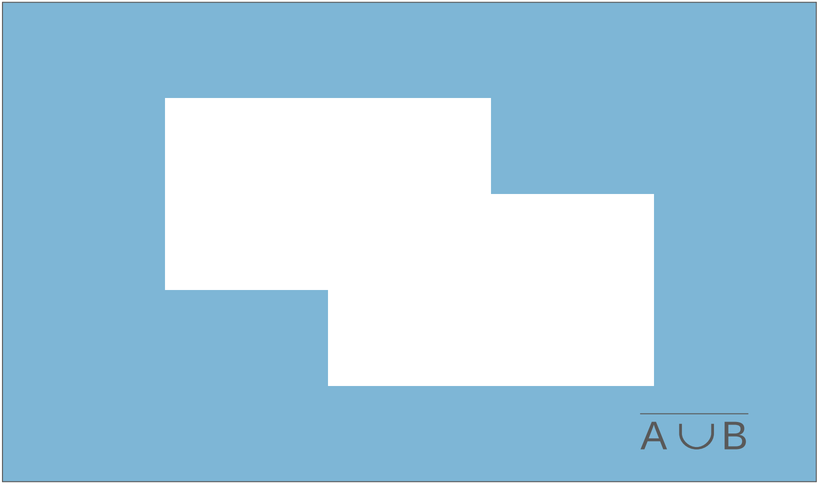


$P(A \cap B) = ?$

Combining events

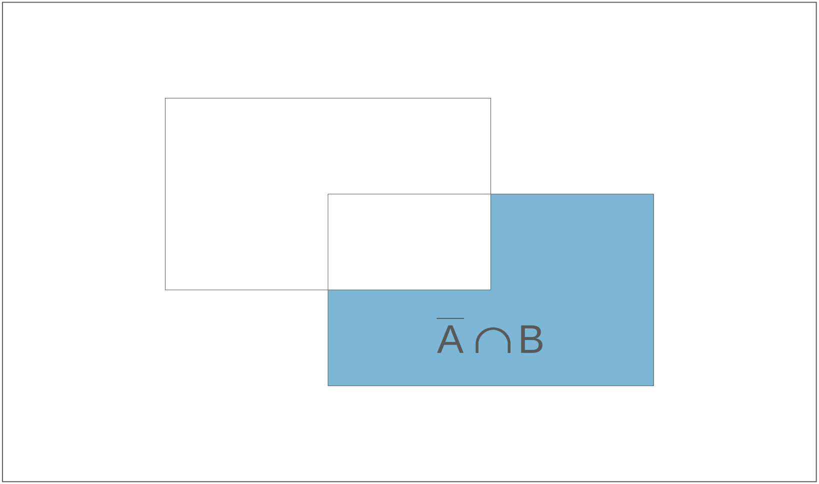


$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



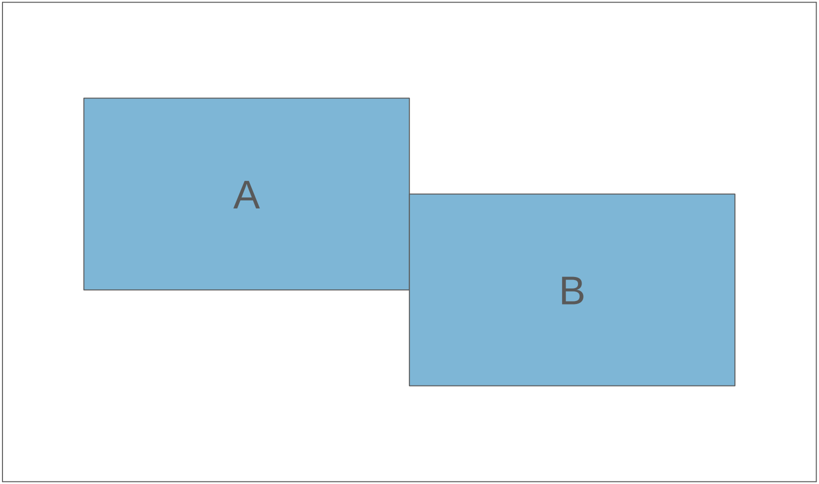
$$P(\overline{A \cup B}) = 1 - P(A \cup B)$$

Combining events



$$P(\overline{A} \cap B) = P(B) - P(A \cap B)$$

Combining events - disjoint events



$A \cap B = \emptyset$; $P(A \cap B) = 0$

Combining events - partition

Consider B

Combining events - partition

Consider B and A_i where $i \in \{1,...,n\}$:

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1. $\forall \{i, j\} \setminus i \neq j$, $P(A_i \cap A_j) = 0$ **pairwise disjoint**

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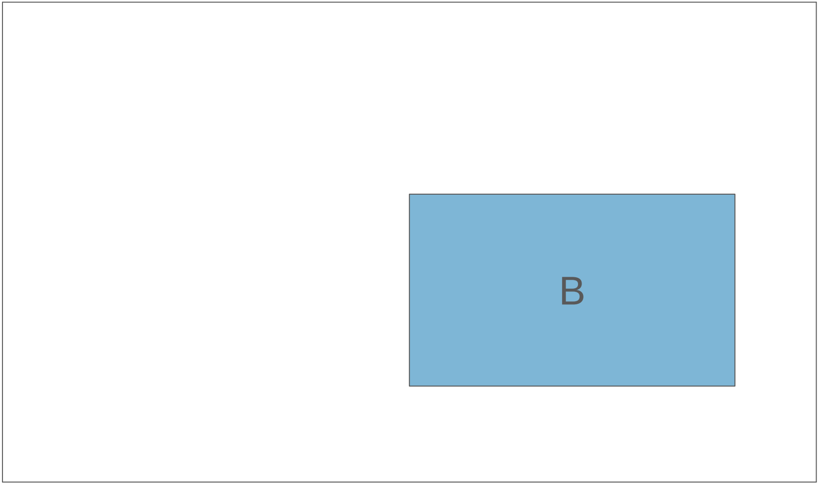
then, the set A_i is a **partition** of B.

Combining events - partition



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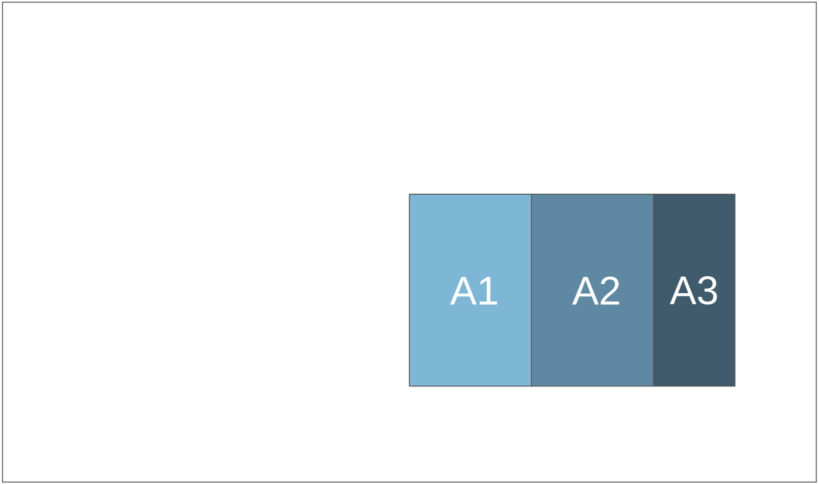

$$P(B) = P(A_1 \cup A_2 \cup A_3)$$

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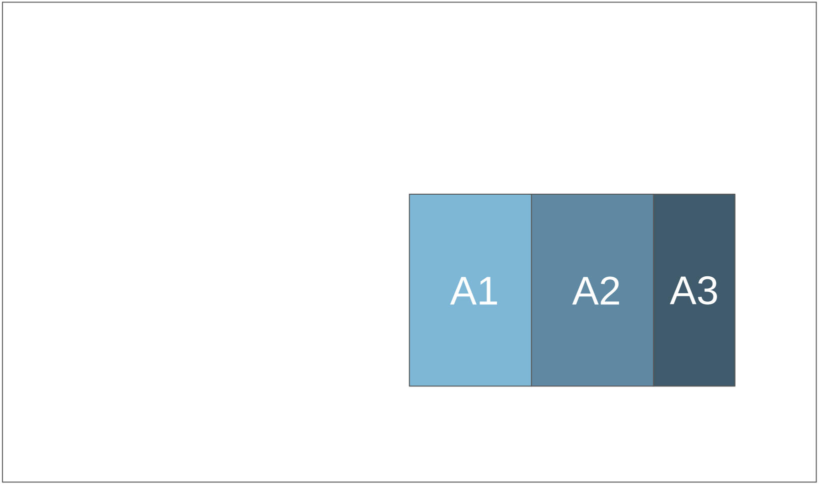
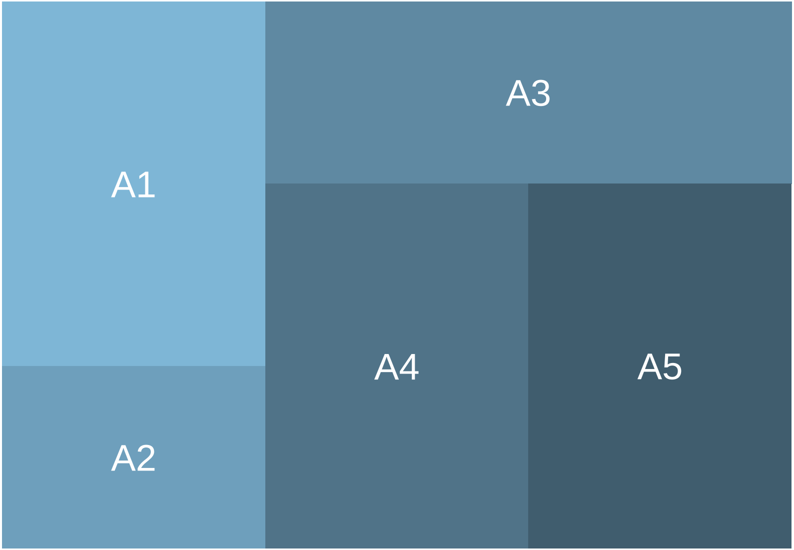


$$P(B) = P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3)$$

Combining events - partition



Combining events - partition

A_i where $i \in \{1, 2, 3, 4, 5\}$ is a partition of Ω

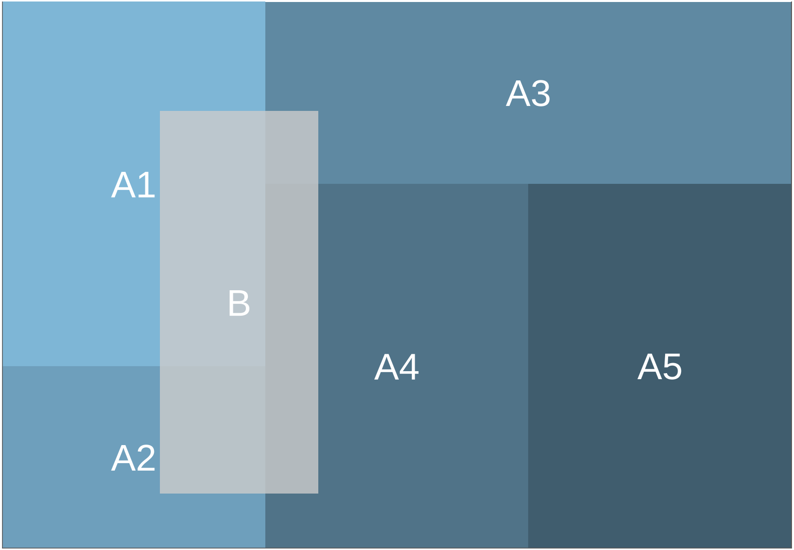
$$\sum_{i}^{5} P(A_i) = 1$$

Combining events - formula (law) of total probability

A_i a partition of Ω and B an event:

$$P(B) = \sum_{i}^{n} P(B \cap A_i)$$

Combining events - formula (law) of total probability



Combining events - formula (law) of total probability

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- Events: $\{$"00", "01", "10", "11"$\}$

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Occurrence of species 1 and species 2 on an island

- Events: {“00”, “01”, “10”, “11”}
- P(”00” ∩ ”01”) = 0
- ”00” ∪ ”01” ∪ ”10” ∪ ”11” = Ω
- {“00”, “01”, “10”, “11”} is a partition of Ω
- $p_{00} + p_{01} + p_{10} + p_{11} = 1$

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Describes a **probability distribution**

Let’s practice 1 (15 min)

PRACTICE 1

- $P(A \cup B \cup C)$

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- $P(A \cup B \cup C)$
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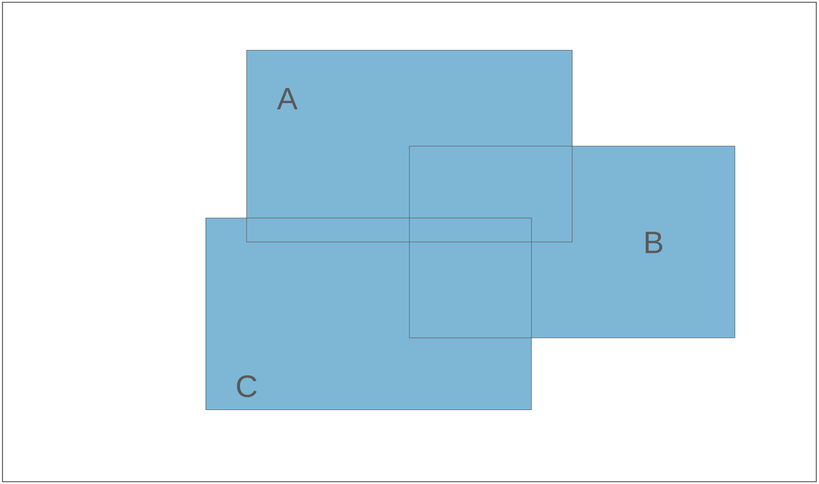
- bonus: how to simulate a dice with a coin?



$P(A \cup B \cup C) = ?$

Let's practice 1 - Elmer, the duck hunter

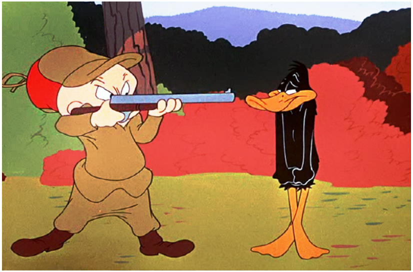
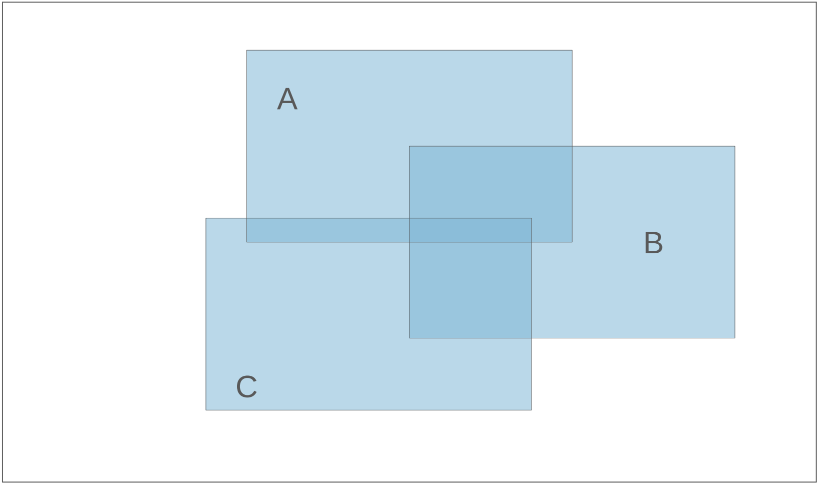


Figure 1: Daffy & Elmer

- Elmer, one bullet, one duck
- Elmer, two bullets, two ducks
- Elmer, two bullets, one duck

Solution 1 - P($A \bigcup B \bigcup C$)



Solution 1 - $P(A \bigcup B \bigcup C)$

- See the Inclusion–exclusion principle article on wikipedia (formule du crible de Poincaré).

Solution 1 - the duck paradigm

- Elmer, one bullet, one duck
- “sucess” (“1”)/ “failure” (“0”)
- P(”sucess”) = p ; P(”failure”) = 1 – p

- Flipping a coin / occurrence of 1 species on an island / shooting a duck

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Random variables

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- *Success*: "Head", "Presence", "one dead duck" $\rightarrow$ **1**

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- Now let X denote a variable such as:
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- Now let X denote a variable such as:
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- Define a random variable + assign a probability distribution.

Random variables and probability distribution

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1. f specifies P for the partition of Ω made of singletons

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2. f defines p_0, p_1, p_2, ..., p_n such as $\sum p_i = 1$

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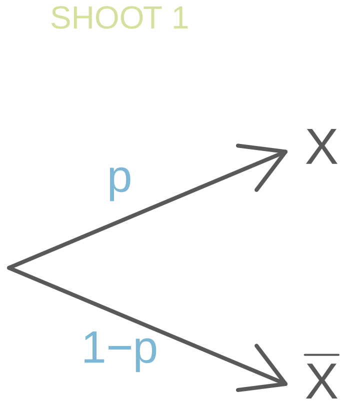
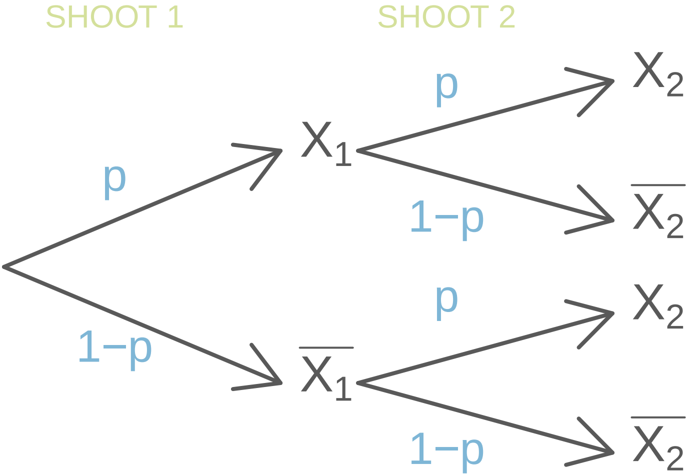
   - Coin P("Head") = P(1) = p; P("Tail") = P(0) = 1 – p
   - Dice P(1) = P(2) = ... = P(6) = 1/6


SHOOT 1
p
1–p
X
$\overline{X}$

Independence - Intuition - 2 ducks



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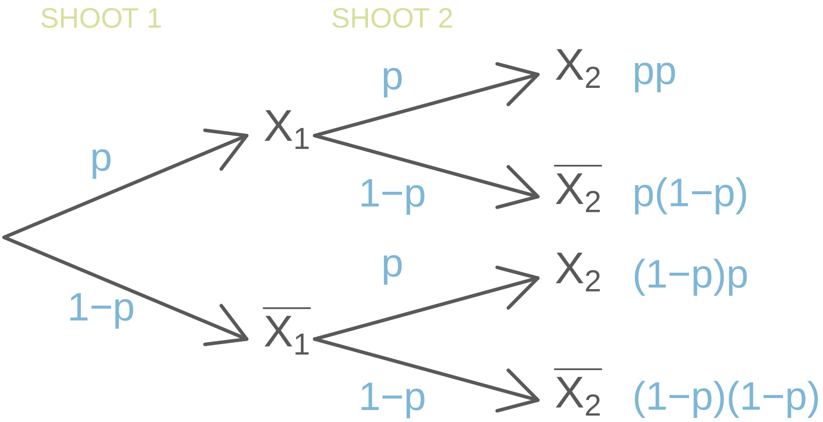
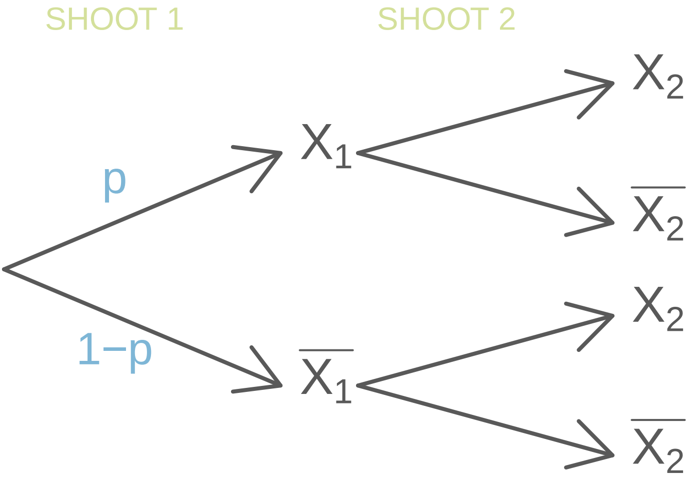
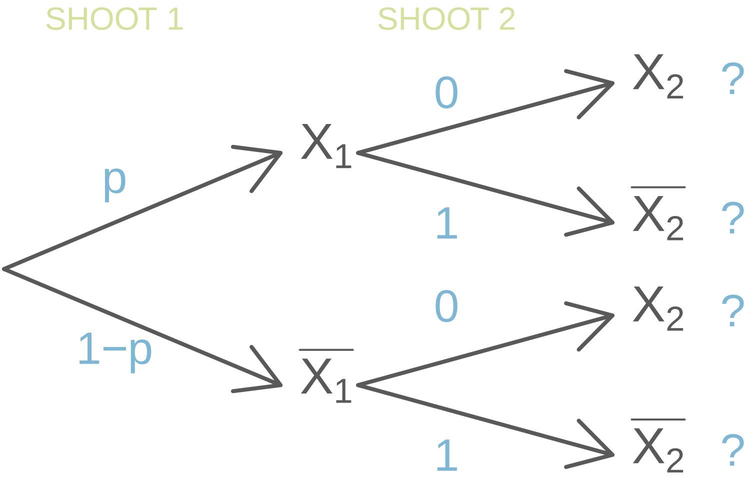


Independence - Intuition - 1 duck



Independence - Intuition - 1 duck



Independence - Definition

Two events are independent iif:

- $P(A \cap B) = P(A)P(B)$

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Remarks:

1. this is an assumption often implicit (notably in statistics)

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Remarks:

1. this is an assumption often implicit (notably in statistics)

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2. events that may not seem independent (intuitively) may be independent according to the definition
3. A and B independent then $P(A \cup B) = P(A) + P(B) - P(A)P(B)$

Let's practice 2 (15 min)

Elmer shoots 3 independent ducks with a success rate of $p = 0.4$

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He now shoots n independent ducks with a success rate of p

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- bonus: solve the **dice problem**

Solution 2

Y = "number of duck Elmer killed",

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Solution 2

$Y =$ "number of duck Elmer killed", $Y \in 0, 1, 2, 3$

$Z =$ "number of failure before first success" $Z = 0, 1, ..., n$

Finite and countably infinite support sets

1. Finite set: $X = \{1, 2, \ldots, n\}$

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- number of species on a given island
- number of failure before the first success
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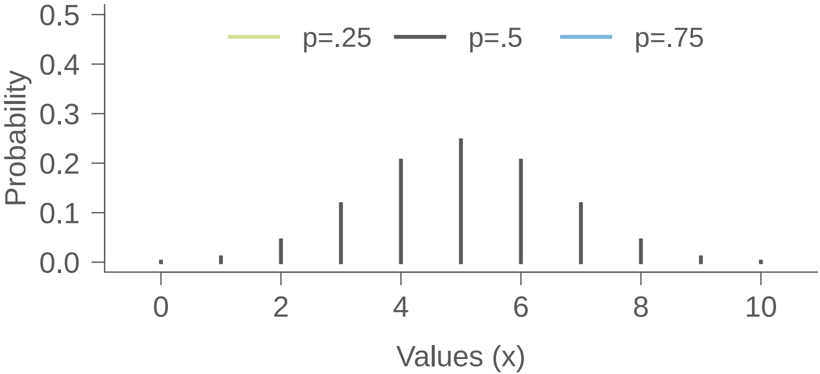
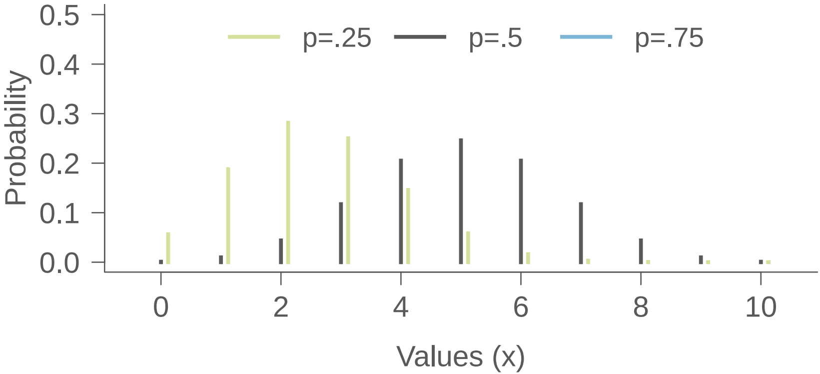
$$\sum_{i}^{+\infty} P(X_i) = 1$$

Binomial distribution *dbinom*



$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

Binomial distribution *dbinom*



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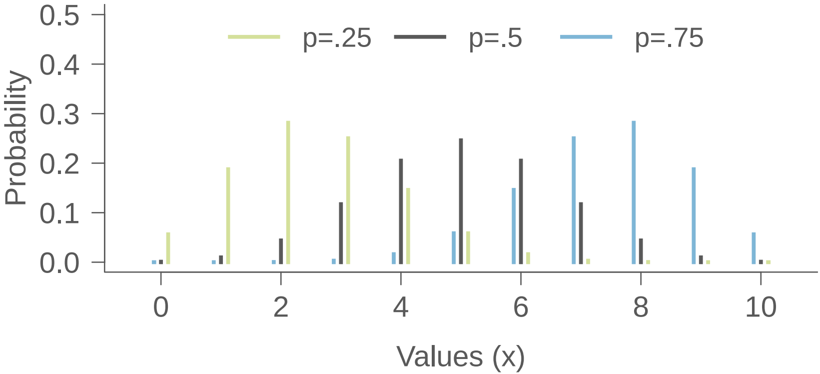
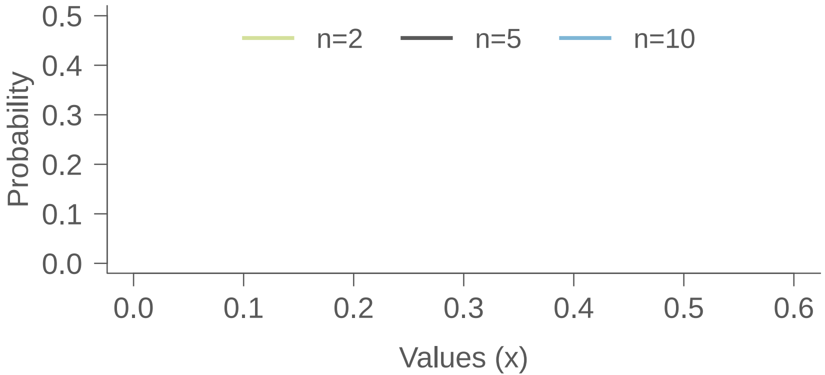


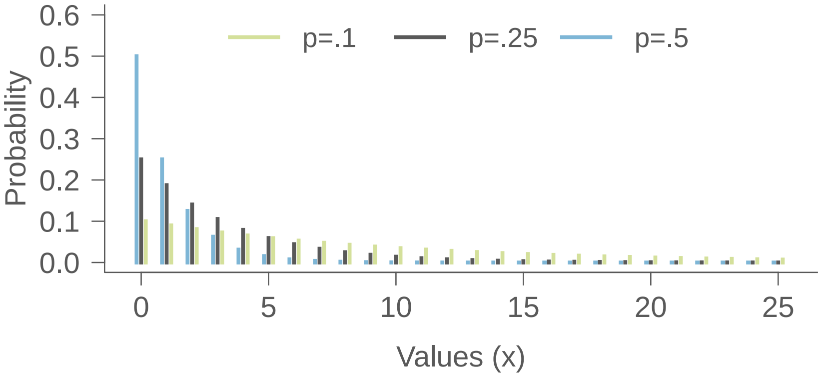
$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Uniform distribution



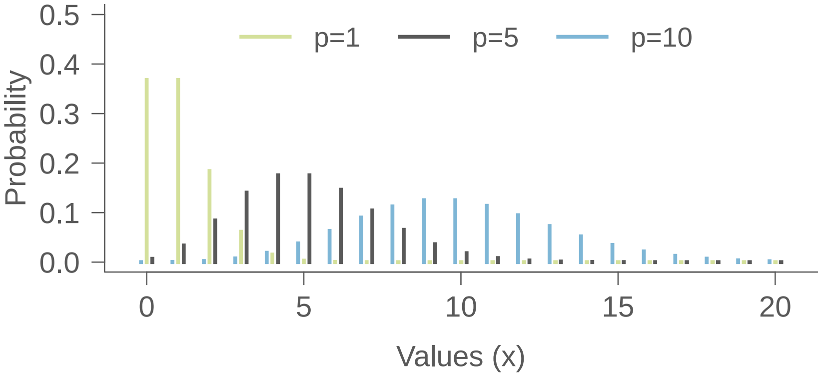
$$P(X = k) = \frac{1}{n}$$

Negative binomial distribution *dnbinom*



$$P(X = k) = p(1 - p)^{k-1}$$

Poisson distribution *dpois*



$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

PAUSE PAUSE PAUSE PAUSE PAUSE PAUSE

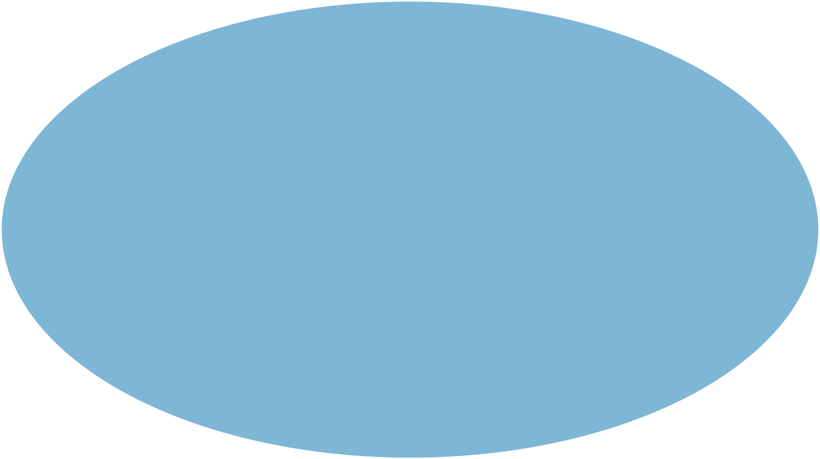
PART 2

Infinite sets

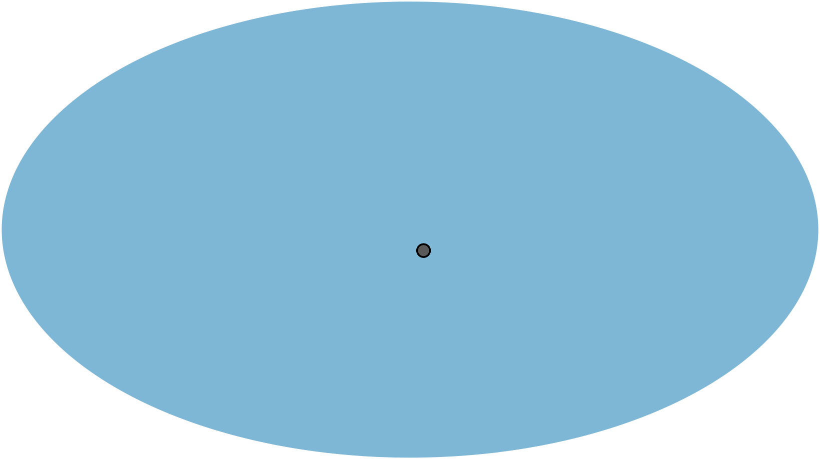
Moments

The Bayes theorem

Infinite set - where is the duck?



Infinite set - where is the duck?



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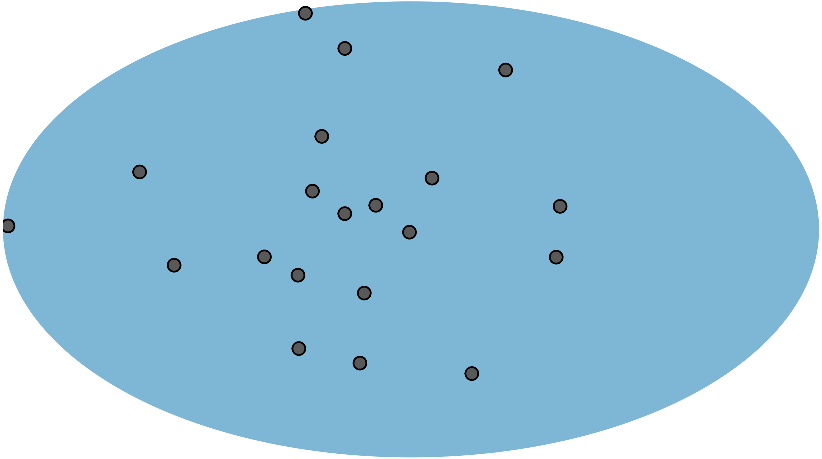

Infinite set - where is the duck?

Let X be the random values x-coordinate

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- values: $x \in [0,10]$

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- $P(X \in [0, 5]) = 0.5$

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Infinite set - where is the duck?

- $P(X = x) = \frac{1}{\infty}$

Infinite set - where is the duck?

- $P(X = x) = \frac{1}{\infty} = 0$

Infinite set - where is the duck?

- $P(X = x) = \frac{1}{\infty} = 0$ but...

Infinite set - where is the duck?

- $P(X = x) = \frac{1}{\infty} = 0$ but...

We need something else!

Infinite set - probability density function (p.d.f)

f is a **p.d.f** iif:

1. defined on [a,b] (a may be $-\infty$ / b may be $+\infty$)

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1. defined on [a,b] (a may be $-\infty$ / b may be $+\infty$)
2. positive
3. regular

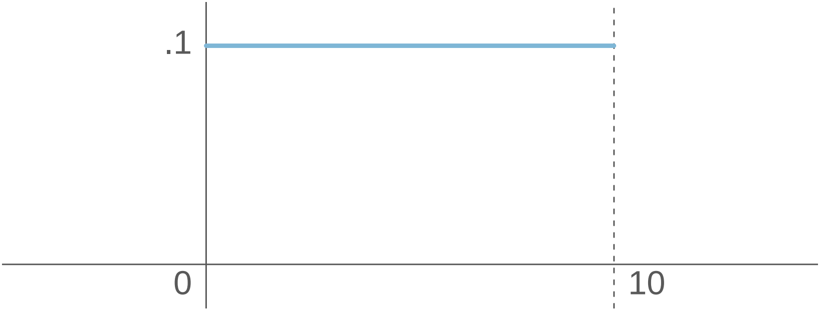
Infinite set - probability density function (p.d.f)

f is a **p.d.f** iif:

1. defined on [a,b] (a may be $-\infty$ / b may be $+\infty$)
2. positive
3. regular
4. and:

$$\int_a^b f(x)dx = 1$$

Infinite set - where's the duck?

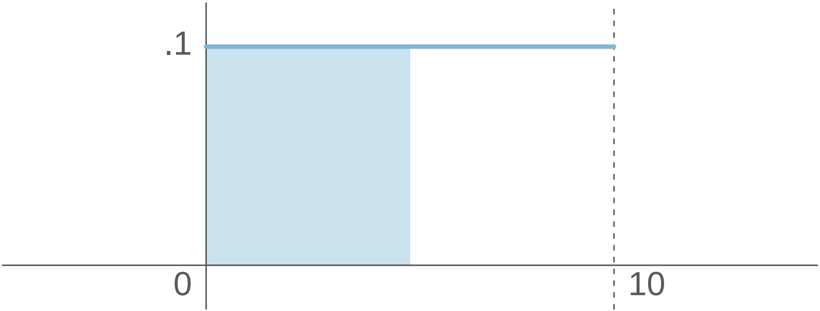


$$\forall\ x \in [0, 10] \quad f(x) = .1 \qquad (\mathcal{U}_{[0,10]})$$

Infinite set - where's the duck?

$$\int_0^{10} f(x)dx = 1$$

Infinite set - where's the duck?



$$\int_0^5 f(x)dx = .5$$

Probability distribution - act 2

Probability distribution function:

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- **probability mass function, p.m.f.**: random variables with a discrete support set (or countable infinite)

Probability distribution - act 2

Probability distribution function:

- **probability mass function, p.m.f.**: random variables with a discrete support set (or countable infinite)
- **probability density function, p.d.f.**: random variables with a infinite support set

Probability distribution - act 2

- f(x)   [x] (pmf or pdf)
- $\int$ f(x)dx $\int$[x]dx

Probability distribution - act 2

- $f(x) \quad [x]$ (pmf or pdf)
- $\int f(x)dx \quad \int [x]dx$

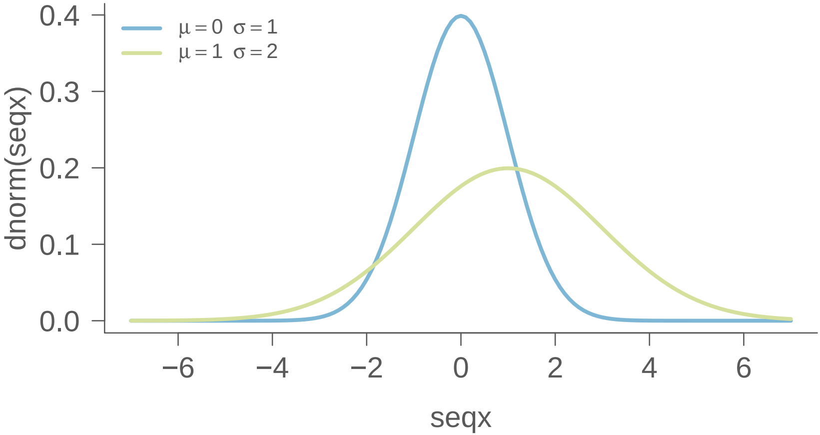
Conditional probability:

- $f(x|y) \quad [x|y]$
- $f(x) = f(x|y)f(y)$
- $f(x) = f(x|y)P(y)$
- $f(x_1)f(x_2)$

Cumulative distribution function (c.d.f.)

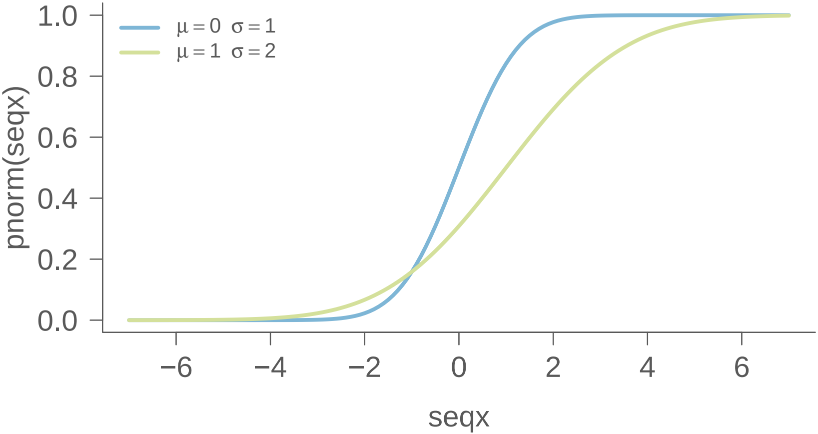
$$F(y) = P(X \leq y) = \int_{-\infty}^{y} f(x)dx$$

Normal distribution - p.d.f. *dnorm*

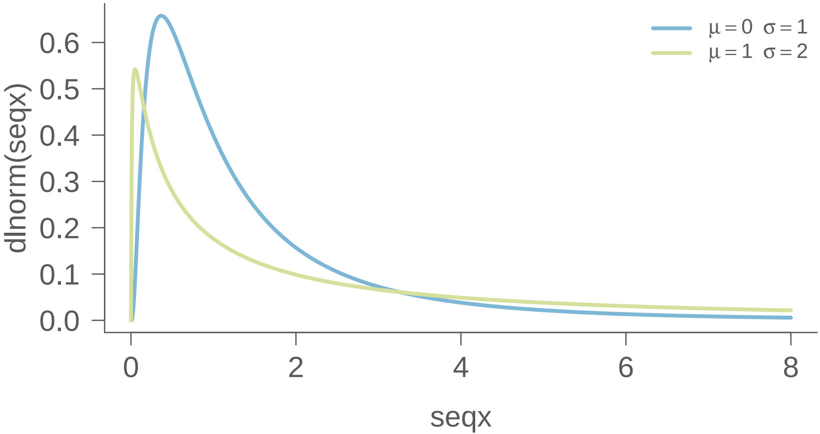


$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Lognormal distribution - c.d.f. *pnorm*

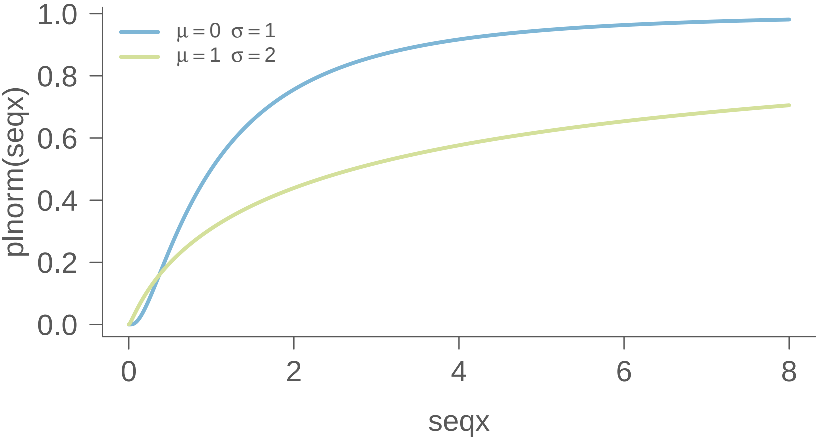


Lognormal distribution - p.d.f. *dlnorm*

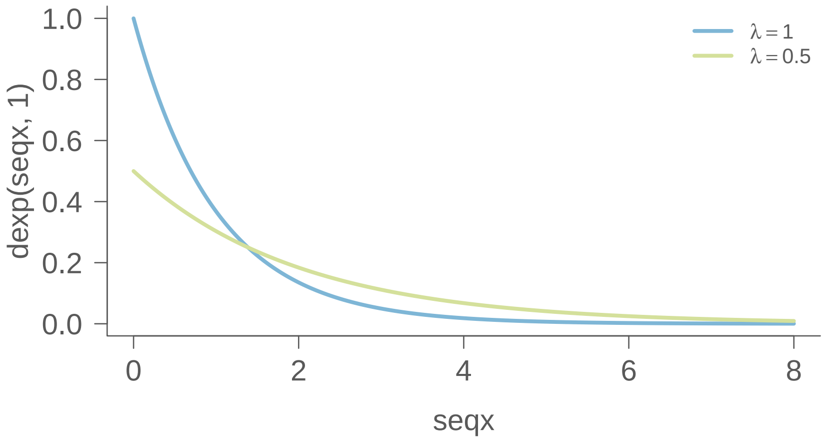


$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2}$$

Lognormal distribution - c.d.f. *plnorm*

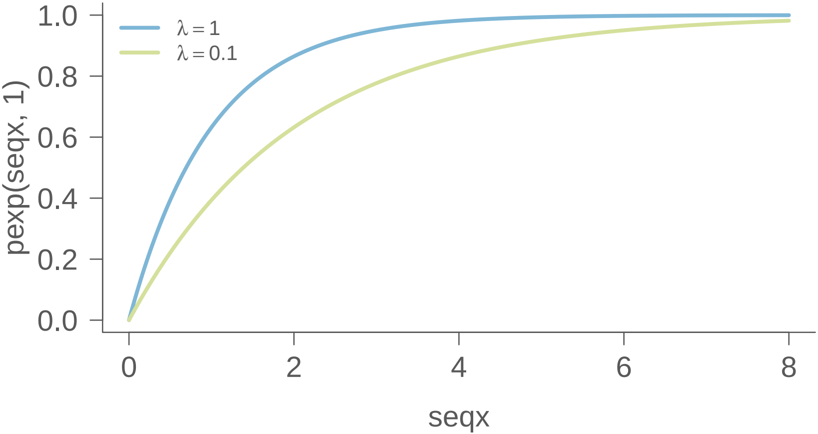


Exponential distribution - p.d.f. *dexp*



$$f(x) = \lambda e^{-\lambda x}$$

Exponential distribution - c.d.f. *pexp*



Practice 3 (10 min)

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- How to model tpon arrival?

- Find the probability that you get started on time?

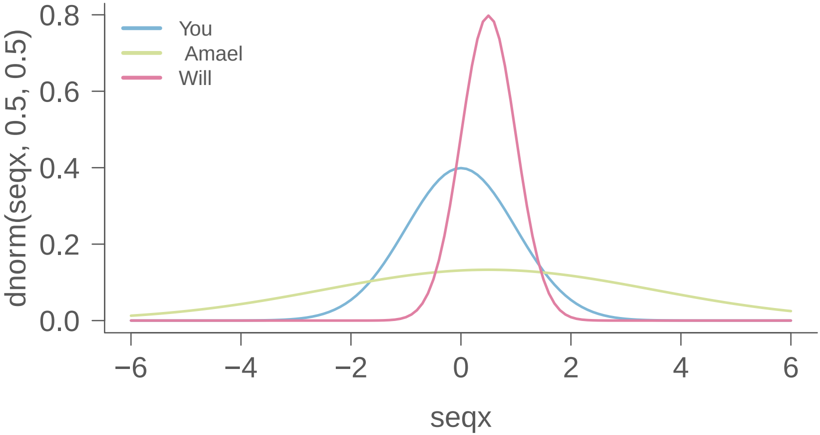
- Find the probability that the meeting is delayed by *at least* half an hour?

Solution 3

1- You: $\mathcal{N}(0, 1)$

2- Amael: $\mathcal{N}(.5, 3)$

3- Will: $\mathcal{N}(.5, .5)$



Solution 3

1. Starting on time
2. Meeting delayed by at least half an hour

Dealing with joint distributions

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1. $P(X \cap Y)$ or $P(X, Y)$

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4. $f(x|y)$, $f(y|x)$

Expectation (*a.k.a* expected value, mean):

$$E(X) = \int xf(x)dx$$

Expectation and moments

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Moment-generating function (MGF) alternative speciation of the distribution.

Quantile α:

$$x_\alpha \quad P(X \leq x_\alpha) = \alpha$$

Quantiles

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Examples:

- median ($\alpha = .5$)
- 1st and 3rd quartile ($\alpha = .25$ $\alpha = .75$)
- 5 / 95 percentile ($\alpha = .05$ $\alpha = .95$)

Quantile α:

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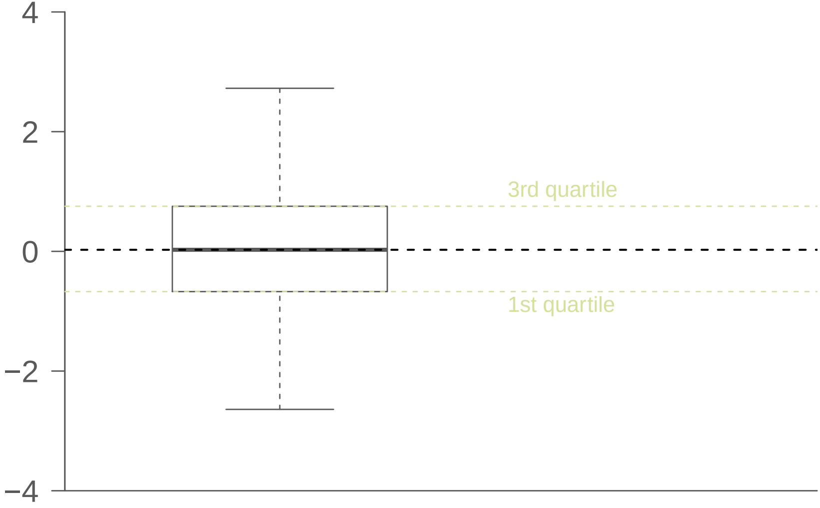
R: `qbinom`, `qpois`, `qnorm`, ...

Quantiles



More about expectation

$$E(g(X)) = \int g(x)f(x)dx$$

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$$Z = X^2 \quad E(Z) = \int x^2 f(x)dx$$

$$Z = \cos(X) \quad E(Z) = \int \cos(x)f(x)dx$$

Expectation / variance

- Binomial: $X : \mathcal{B}(n, p)$

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$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$E(X) = \sum_k^n kP(X = k) = \sum_k^n k \binom{n}{k} p^k (1 - p)^{n-k}$$

$$E(X) = np$$

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$$E(X) = np$$

$$V(X) = npq$$

Example Expectation / variance

- Poisson $\mathcal{P}(\lambda)$: $E(X) = \lambda$; $V(X) = \lambda$
- Binomial negative: $\mathcal{NB}(r, p)$: $E(X) = \frac{(1-p)r}{p}$; $V(X) = \frac{(1-p)r}{(p)^2}$
- Binomial negative: $\mathcal{NB}(1, p)$: $E(X) = \frac{(1-p)}{p}$; $V(X) = \frac{(1-p)}{(p)^2}$
- Normal $\mathcal{N}(\mu, \sigma)$: $E(X) = \mu$; $V(X) = \sigma^2$
- Exponential $\mathcal{E}(\lambda)$: $E(X) = \lambda$; $V(X) = \lambda^2$

Example Expectation / variance

Notation	$\mathcal{N}(\mu, \sigma^2)$
Parameters	$\mu \in \mathbb{R}$ — mean (location) $\sigma^2 > 0$ — variance (squared scale)
Support	$x \in \mathbb{R}$
PDF	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
CDF	$\frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$
Quantile	$\mu + \sigma\sqrt{2}\operatorname{erf}^{-1}(2F-1)$
Mean	μ
Median	μ
Mode	μ
Variance	σ^2
Skewness	0
Ex. kurtosis	0

Figure 2: Normal distribution's properties on Wikipedia

Let's practice 4 (15 min)

Elmer and the frightening question!

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Elmer and the frightening question!

- Elmer's success rate is p

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- Elmer's success rate is p
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- "Should Elmer better stop hunting?"

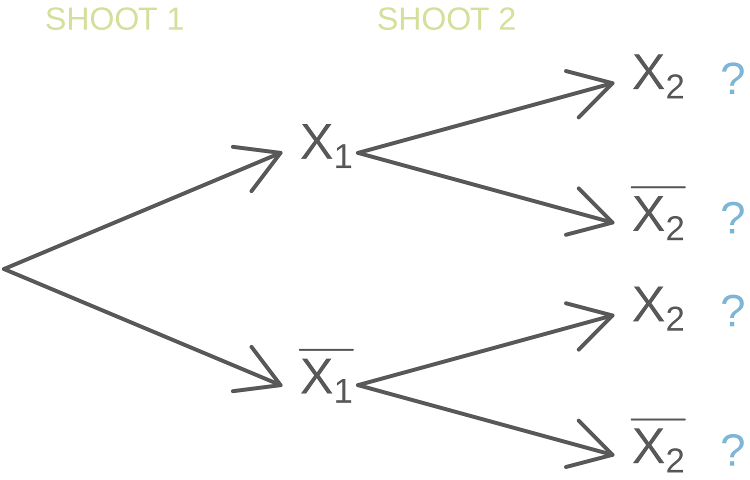
Let's practice 4 (15 min)

Elmer and the frightening question!

- Elmer's success rate is p
- a bullet is 3$
- a duck of the same quality is 60$
- "Should Elmer better stop hunting?"
- Find p_{sh} the success rate below which Elmer should better stay at home?

Solution 4

Independence act 2



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Let's A and B be two events, the conditional probability $P(A|B)$ is defined as:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

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$$P(X_1 = 1|X_2 = 1) = \frac{P(X_1 = 1 \cap X_2 = 1)}{P(X_2 = 1)}$$

Independence act 2

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consequently:

$$P(A \cap B) = P(A|B)P(B)$$

Independence act 2

Independence:

$$P(A|B) = P(A)$$

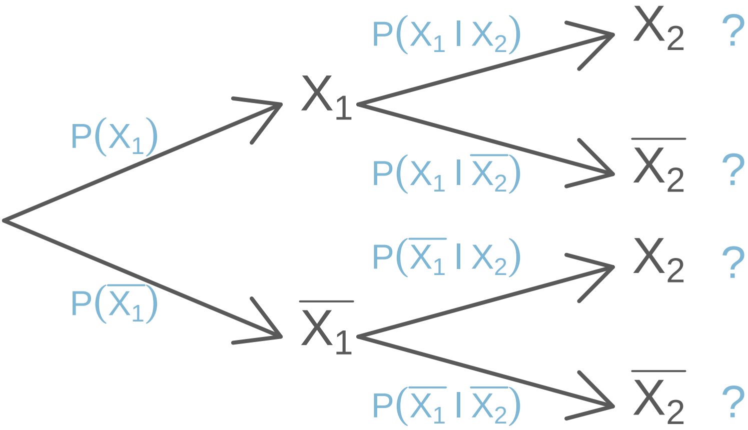
Independence act 2

Independence:

$$P(A|B) = P(A)$$

$$P(A \cap B) = P(A|B)P(B)$$

Independence act 2



Bayes theorem

Bayes theorem

$$P(A \cap B) = P(B \cap A)$$

$$P(A \cap B) = P(B \cap A)$$

$$P(A|B)P(B) = P(B|A)P(A)$$

$$P(A \cap B) = P(B \cap A)$$

$$P(A|B)P(B) = P(B|A)P(A)$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Bayes theorem

"Given the number of times in which an unknown event has happened and failed: Required the chance that the probability of its (specific event) happening in a single trial lies somewhere between any two degrees of probability that can be named."

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Proposition 3:

"The probability that two subsequent events will both happen is a ratio compounded of the probability of the 1st, and the probability of the 2nd on supposition the 1st happens."

"Given the number of times in which an unknown event has happened and failed: Required the chance that the probability of its (specific event) happening in a single trial lies somewhere between any two degrees of probability that can be named."

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An Essay Towards Solving a Problem in the Doctrine of Chances

Proposition 5:

> *"If there be two subsequent events, the probability of the 2nd b/N and the probability both together P/N, and it being first discovered that the 2nd event has happened, from hence I guess that the 1st event has also happened, the probability I am in the right is P/b"*

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Bayes theorem

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Practice 5 - Are you infected? (20 min)

- Prevalence is π (0.01)

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- You take the test, it is negative, are you infected?
- bonus: build a function to answer the questions above for any parameters' value.

Solution 5

Let's use 2 random variables:

- $X = 1$ ("sick"); $X = 0$ ("sane")
- $T = 1$ ("test positive"); $T = 0$ ("test negative")

LUNCH LUNCH LUNCH LUNCH LUNCH

PART 3

Let's practice more

Practice 6 - Elmer is back (25 min)

- Elmer's precision decreases as distance increases

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3. Elmer brings 10 bullets, what's the probability he'll have a nice diner?

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- bonus: solve the **division problem**

Solution 6

Practice 7 - Elmer... the truth (25 min)

`val1.csv` (or `val1.Rds`) are the results of 1000 shoots Elmer took.

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3. Create a function that computes $P(p|X)$ for any value of p.
4. We have a new set of data `val2.csv` or `val2.Rds`, what should you do?
5. bonus: 1-3 including the distance (see `val3.csv` or `val3.Rds`)
6. bonus 2: Answer Bayes' original question

Solution 7

Let's step back

What do we do when we do statistics? (simple case)

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- The **distribution** is given by θ (*i.e.* $\mathcal{N}(\theta)$ where $\theta = (\mu, \sigma)$)
- We try to find out θ 's value(s) given x_i : **inference**

Let's take a step back

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- Then we assess the goodness of our estimation : IC / tests
- Bayesian framework offers few other possibilities.

Why normal, why?

Central limit theorem:

$$X_i, i \in 1, 2, ..., n$$

i.i.d. $\mathcal{L}(\theta)$,

$$\frac{X_i - \mu}{\sigma} \rightarrow N(\mu, \sigma)$$

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Poincaré:

> *"Tout le mode croit à la loi normale : les physiciens parcequ'ils pensent que les mathématiciens l'ont démontrée et les mathématiciens parcequ'ils croient qu'elle a été vérifiée par les physiciens."*

To be continued